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$L_1(G, A)$-multipliers

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1. Introduction

Throughout this paper, G will be a locally compact group and A a complex Banach algebra.

In [9] MING-KAM CHAN characterizes the $L_1(G, A)$-multipliers for algebras having a weak (hence a strong) bounded approximate identity.

By the present work, we prove that the characterizations remain true even in case A doesn't possess such an approximate identity, using the fact that any Banach algebra is contained in a Banach algebra with identity. Next, we enter upon the situation (not considered in [9]) where G is compact, non abelian. Doing this, we are induced to extend the notion of Fourier—Stieltjes transform of a vector measure.

2. Terminology

2.1. Vector measures. Let S be a locally compact (Hausdorff) space and E, a real or complex normed space. Denote by $\mathscr{K}(S, E)$ the vector space over the same field as E of all continuous functions on S into E, having compact supports, and write $\mathscr{K}(S)$ for $\mathscr{K}(S, \mathbf{R})$.

Let F be a real Banach space. By definition, an F-valued vector measure on S is a linear mapping m: $\mathscr{K}(S) \to F$ such that, for every compact set $K \subset S$, there exists a non negative constant α_K and $\|m(f)\| \leqq \alpha_K \sup_{t \in S} |f(t)|$ for every function f with support in K. $m(f)$ is also written

$$\int_S f\,dm, \quad \int_S f(t)\,dm(t) \quad \text{or} \quad \int f(t)\,dm(t).$$

(See [2], chap VI, § 2, no. 1.)

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Suppose now F is complex and consider the underlying real Banach space F_0. Then any **R**-linear mapping: $\mathscr{K}(S)\to F_0$ extends uniquely to a **C**-linear mapping: $\mathscr{K}(S, \mathbf{C})\to F$. As such, we shall always identify every measure m: $\mathscr{K}(S)\to F_0$ with the corresponding linear mapping (still denoted m) from $\mathscr{K}(S, \mathbf{C})$ into F, and shall call again vector measure, any linear mapping m: $\mathscr{K}(S, \mathbf{C})\to F$ whose restriction to $\mathscr{K}(S)$ is a measure (into F_0).

2.2. Bounded measures. A vector measure is said to be dominated if there exists a positive measure μ such that $\left\|\int f(t)\,dm(t)\right\| \leqq \int |f(t)|\,d\mu(t)$, $f\in\mathscr{K}(S)$.

If m is dominated, then there exists a smallest positive measure $|m|$ called the modulus or the variation of m, that dominates it. A positive measure is said to be bounded if it is continuous in the uniform norm topology of $\mathscr{K}(S)$.

A vector measure is said to be bounded if it is dominated by a bounded positive measure. It is clear that m is bounded if and only if $|m|$ is bounded ([3], § 3).

2.3. Integration. Assume μ is a positive measure on S and E is a Banach space. For a function f: $S\to E$, put

$$N_p(f) = \Big(\int^{*} (\|f(t)\|_E)^p\, d\mu(t)\Big)^{1/p}, \quad 1 \leqq p \lneqq \infty$$

where $\int^{*}$ designates the upper integral ([2], chap. IV, § 1, no. 3).

$$N_\infty(f) = \inf\{\alpha\colon \|f(t)\|_E \leqq \alpha, \ \ \mu\text{-almost everywhere}\}.$$

The vector space (over the same field as E) of all μ-measureable functions f: $S\to E$ such that $N_p(f)<\infty$ is denoted by $\mathscr{L}_p(S, \mu, E)$ or $\mathscr{L}_p(S, E)$... and the corresponding quotient space $\mathscr{L}_p(S, \mu, E)/\mathscr{N}$ with respect to the closed subspace of the negligible functions, by $L_p(S, \mu, E)$ or $L_p(S, E)$. The seminorm N_p induces a norm $\|\cdot\|_{pE}$ on $L_p(S, E)$ which becomes a Banach space. In the sequel, we shall write f for the class $[f]$ as it is usually done.

With the positive measure μ on S is uniquely associated a continuous linear mapping n: $\mathscr{K}(S, E)\to E$ given by the equation

$$n(x\psi) = x\mu(\psi), \quad x\in E, \quad \psi\in\mathscr{K}(S)$$

([3], 2.11). Since $\mathscr{K}(S, E)$ is dense in $\mathscr{L}_1(S, E)$, n has an extension (still called n) to $\mathscr{L}_1(S, E)$. The integral of $f\in\mathscr{L}_1(S, E)$ with respect to μ is the value $n(f)$ denoted:

$$\int_S f(t)\,d\mu(t).$$

(It belongs to E.)

Now, let m be a dominated measure with values in a Banach space F. Then the space $\mathscr{L}_p(S, m, E)$ is by definition the space $\mathscr{L}_p(S, |m|, E)$. We associate with

m in a unique manner a continuous linear mapping (still called m) from $\mathcal{L}_1(S, m, E)$ into a Banach space D, provided there exists a continuous bilinear mapping: $E\times F\to D$. The corresponding integral $\int f(t)\,dm(t)$ belonging to D, is the value $m(f)$ ([3], 8.61. or [2], chap. VI, § 2, no. 7). If $f\in\mathcal{L}_1(S, \mathbf{R})$, put $D=F$ and $E=\mathbf{R}$, and the required bilinear mapping is the multiplication by real numbers: $\mathbf{R}\times F\to F$.

2.4. Convolution. The space $M_1(A)=M_1(G, A)$ of all bounded A-valued measures on G is a Banach algebra with the norm $\|m\|=\int \chi_G\, d|m|$ where χ_G is the characteristic function of G, and the convolution

$$m*n(f)=\int\Big(\int f(st)\,dm(s)\Big)\,dn(t),\quad f\in\mathcal{K}(G)\quad\text{and}\quad m, n\in M_1(A),$$

$$\text{shortly written}\quad \iint f(st)\,dm(s)\,dn(t).$$

Let λ be the left Haar measure on G. Identifying $f\in L_1(G, \lambda, A)$ with the bounded measure $f\lambda$ defined by

$$f\lambda(g)=\int f(t)g(t)\,d\lambda(t),\quad g\in\mathcal{K}(G)$$

then the functions

$$t\to f*g(t)=\int f(s)g(s^{-1}t)\,ds,\quad f, g\in L_1(G, A),\quad ds=d\lambda(s),$$

$$t\to m*f(t)=\int f(s^{-1}t)\,dm(s),\quad m\in M_1(A),\quad f\in L_1(G, A),$$

and

$$t\to f*m(t)=\int f(ts^{-1})\Delta(s^{-1})\,dm(s),\quad f\in L_1(G, A),\quad m\in M_1(A),$$

(where Δ is the modular function of G), belong to $L_1(G, A)$. Consequently, $L_1(G, A)$ appears as a two-sided ideal of $M_1(A)$.

2.4.1. Lemma. *Let τ_s, $s\in G$, be the right translation: $\tau_s f(t)=f(ts^{-1})$. Then*

(i) $\tau_s(f*g)=f*\tau_s g=(\Delta(s)\tau_{s^{-1}}f)*g,\quad f, g\in L_1(G, A),$

(ii) $\tau_s(m*g)=m*\tau_s g,\quad m\in M_1(A)$ *and* $g\in L_1(G, A)$.

The proof is straightforward.

2.4.2. Lemma. *If m is bounded and $m*g=0$ or $g*m=0$ for every $g\in\mathcal{K}(G)$ or for every $g\in\mathcal{K}(G, A)$, then $m=0$.*

See [3], 24.35 for the proof.

From now on, we shall write dt for $d\lambda(t)$.

3. $L_1(G, A)$-multipliers for general locally compact groups

3.1. Definition. A left $L_1(G, A)$-multiplier is a continuous linear operator $T\colon L_1(G, A)\to L_1(G, A)$ such that

$$(3.1.1)\qquad \tau_s T = T\tau_s,\quad s\in G \quad\text{and}\quad T(xf) = xT(f),\quad x\in A,\quad f\in L_1(G, A).$$

A right $L_1(G, A)$-multiplier is respectively defined. Now, any result true for left multipliers has its analogue for right multipliers. Therefore we are going to study the left multipliers only, and omit the word "left" and sometimes the symbol "$L_1(G, A)$". By [6], it is not necessary to include continuity in the definition of multipliers. We do it only to avoid superfluous discussions.

We are going to use in the proof of the next theorem a number of facts that we want to point out now.

3.2. Extension of $L_1(G, A)$-multipliers to $L_1(G, \overset{*}{A})$. In the sequel, $\overset{*}{A}$ will be the Banach algebra obtained by joining an identity e to A. We recall that A is a two sided maximal ideal in $\overset{*}{A}$.

3.2.1. Lemma. *The space $L_1(G, \overset{*}{A})$ is norm isomorphic to the direct sum $L_1(G, A)\oplus eL_1(G)$.*

Proof. It is known that $\overset{*}{A}\cong A\oplus e\mathbf{C}$. Hence every $F\in L_1(G, \overset{*}{A})$ may be represented as $f=f+e\varphi$, $f\in L_1(G, A)$ and $\varphi\in L_1(G)$. To see it, put $f=P\circ F$ and $e\varphi=(I-P)\circ F$ where P is the projection operator: $\overset{*}{A}\to A$. Indeed, f and φ are integrable ([4] p. 480 and [3], 8.3).

3.2.2. Lemma. *Let T be an $L_1(G, A)$-multiplier. Then T is extendable to an $L_1(G, \overset{*}{A})$-multiplier $\overset{*}{T}$ and there exists a linear operator*

$$\tau\colon L_1(G)\to L_1(G)\quad \text{such that}\quad T(x\varphi) = x\tau(\varphi),\quad x\in A\quad \text{and}\quad \varphi\in L_1(G).$$

Proof. The condition $T(xf)=xT(f)$ in (3.1.1) shows that T is an A-module homomorphism on $L_1(G, A)$; hence according to [2], § 1, no. 1, T has an extension $\overset{*}{T}$ which is $\mathbf{C}$- and A-linear on $L_1(G, \overset{*}{A})$. Thus, for $X=x+e\xi$ in $\overset{*}{A}$ and $F= =f+e\varphi$ in $L_1(G, \overset{*}{A})$ we have:

$$\overset{*}{T}(XF) = \overset{*}{T}(xf+\xi f+x\varphi+e\xi\varphi) = x\overset{*}{T}(f)+\xi\overset{*}{T}(f)+x\overset{*}{T}(e\varphi)+\xi\overset{*}{T}(e\varphi).$$

Put $\overset{*}{\tau}=$the restriction of $\overset{*}{T}$ to $eL_1(G)$. Since $\overset{*}{T}|_{L_1(G, A)}=T$, it is clear that

$$\overset{*}{T}(XF) = xT(f)+\xi T(f)+x\overset{*}{\tau}(e\varphi)+\xi\overset{*}{\tau}(e\varphi) = (x+e\xi)T(f)+(x+e\xi)\overset{*}{\tau}(e\varphi) = X\overset{*}{T}(F).$$

The definition of $\overset{*}{\tau}$ proves that $\overset{*}{\tau}(e\varphi)=e\tau(\varphi)$ for some linear mapping $\tau: L_1(G)\to L_1(G)$. Hence $\overset{*}{T}=T+e\tau$.

For $x\in A$ and $\varphi\in L_1(G)$, $x\varphi\in L_1(G, A)$; then

$$T(x\varphi) = \overset{*}{T}(xe\varphi) = x\overset{*}{\tau}(e\varphi) = x\tau(\varphi).$$

It is easy to deduce that τ is continuous, $\|\tau\|\leqq\|T\|$ and $\tau_S\tau=\tau\tau_S$. Now,

$$\|\overset{*}{T}(F)\|_{1\overset{*}{A}} \leqq \|T(f)\|_{1A}+\|\tau(\varphi)\|_1 \leqq \|T\|\|f\|_{1A}+\|\tau\|\|\varphi\|_1 \leqq$$
$$\leqq \|T\|(\|f\|_{1A}+\|\varphi\|_1) = \|T\|\|F\|_{1\overset{*}{A}}.$$

Hence $\overset{*}{T}$ is continuous. Finally

$$\tau_s\overset{*}{T}(F) = \tau_sT(f)+e\tau_s\tau(\varphi) = T(\tau_sf)+e\tau(\tau_s\varphi) = \overset{*}{T}\tau_s(F).$$

Therefore, $\overset{*}{T}$ is an $L_1(G, \overset{*}{A})$-multiplier.

Now, here is the first main theorem:

3.3. Theorem. *Let T be a continuous linear operator from $L_1(G, A)$ into $L_1(G, A)$. Then the following statements are equivalent:*

(3.1.1) $\quad T\tau_s = \tau_sT,\ s\in G$ *and* $T(xf) = xT(f),\ x\in A,\ f\in L_1(G, A)$,

(3.3.1) $\quad T(f*\mu) = T(f)*\mu,\ f\in L_1(G, A),\ \mu\in M_1(G, \mathbf{C})$,

(3.3.2) $\quad T(f*g) = T(f)*g,\ f$ *and* $g\in L_1(G, A)$,

(3.3.3) $\quad T(f) = m*f$ *for some* $m\in M_1(G, \overset{*}{A})$.

Proof. (a) Assume (3.1.1) and denote by A' the topological dual of A. Using [3] Corollary 14.21, we claim that, if $t\to\langle f(t), g(t)\rangle$ is negligible for every $g\in L_\infty(G, A')$ then f, $f\in L_1(G, A)$ is negligible because the assertion is true for functions of the form $x'\varphi$, $x'\in A'$ and $\varphi\in\mathcal{K}(G)$, which belong to $L_\infty(G, A')$.

We know that $L_\infty(G, A')\subset L_1(G, A)'$. Let $\mu\in M_1(G, \mathbf{C})$ and T' be the ajoint of T. We have, for $g\in L_\infty(G, A')$,

$$\int\langle T(f*\mu)(t), g(t)\rangle\, dt = \int\Big\langle\int f(ts^{-1})\Delta(s^{-1})\, d\mu(s), T'(g)(t)\Big\rangle\, dt =$$
$$= \iint\langle f(ts^{-1}), T'(g)(t)\rangle\Delta(s^{-1})\, d\mu(s)\, dt.$$

Applying Fubini's theorem, we have:

$$\iint\langle f(ts^{-1}), T'(g)(t)\rangle\Delta(s^{-1})\, d\mu(s)\, dt = \iint\langle f(ts^{-1}), T'(g)(t)\rangle\Delta(s^{-1})\, dt\, d\mu(s) =$$
$$= \iint\langle T(f)(ts^{-1}), g(t)\rangle\Delta(s^{-1})\, dt\, d\mu(s)$$

according to (3.1.1). We apply once again the Fubini's theorem.

$$\iint \langle T(f)(ts^{-1}), g(t)\rangle \Delta(s^{-1})\, dt\, d\mu(s) = \int \Big\langle \int T(f)(ts^{-1})\Delta(s^{-1})\, d\mu(s), g(t)\Big\rangle\, dt =$$
$$= \int \big\langle (T(f)*\mu)(t), g(t)\big\rangle\, dt.$$

Hence $T(f*\mu)=T(f)*\mu$. Therefore (3.1.1)⇒(3.3.1).

(b) Assume (3.3.1). We know that the projective tensor product $L_1(G)\otimes_\pi A$ is dense in $L_1(G, A)$. Then it suffices to prove (3.3.2) for $g=x\varphi$, $x\in A$ and $\varphi\in L_1(G)$.

Now, putting $\mu=\varphi\lambda$ in (3.3.1), $T(f*x\varphi)=T\big(x(f*\varphi)\big)=xT(f)*\varphi=T(f)*x\varphi$. Thus (3.3.1)⇒(3.3.2).

(c) Suppose (3.3.2). Then, for $f\in L_1(G, A)$ and $g\in \mathscr{K}(G, A)$ we have

$$T\tau_s(f)*g = T\big(\tau_s(f)\big)*g = T\big(\Delta(s^{-1})\tau_{s^{-1}}(f*g)\big) = \Delta(s^{-1})T\big(f*\tau_{s^{-1}}(g)\big) =$$
$$= \Delta(s^{-1})\big(T(f)*\tau_{s^{-1}}(g)\big) = \Delta(s^{-1})\tau_{s^{-1}}\big(T(f)*g\big) = \tau_s T(f)*g.$$

Consequently $T\tau_s(f)=\tau_s T(f)$, $f\in L_1(G, A)$ (Lemma 2.4.2) and hence $T\tau_s=\tau_s T$, $s\in G$. Moreover, for $x\in A$, the equalities

$$T(xf)*g = T(xf*g) = T(f*xg) = T(f)*xg = xT(f)*g$$

hold.

Therefore (3.1.1) obtains and hence (3.3.2)⇒(3.1.1). We deduce that (3.1.1), (3.3.1) and (3.3.2) are equivalent. To show that (3.3.3) is equivalent to them, let us suppose (3.1.1). Since $\overset{*}{A}$ has an identity and $L_1(G)$, an approximate identity, $L_1(G, \overset{*}{A})$ possesses an approximate identity. By Lemma 3.2.2., T is extendable to an $L_1(G, \overset{*}{A})$-multiplier $\overset{*}{T}$. Applying [9], Theorem 4 (ii) and results on page 181 § 2 to $\overset{*}{T}$, we conclude that there exists a vector measure m, $m\in M_1(G, \overset{*}{A})$ such that

$$\overset{*}{T}(F) = m*F,\quad F\in L_1(G, \overset{*}{A}),$$

identifying $\overset{*}{A}$ with its canonical image in its second conjugate space $\overset{*}{A}''$ (see also [9], page 186, Remark (2)).

Finally, for $F\in L_1(G, A)$, $T(f)=\overset{*}{T}(f)=m*f$, which is (3.3.3).

Conversely, by Lemma 2.4.1. (ii) and the fact that $m*xf=x(m*f)$, $x\in A$, the implication (3.3.3)⇒(3.1.1) is clear. This ends the proof of the theorem.

3.4. **Remark.** (i) In the proof of Lemma 3.2.2., we saw that the extension $\overset{*}{T}$ of T has the form $\overset{*}{T}=T+e\tau$ where τ is related to T by the equation

$$T(x\varphi) = x\tau(\varphi),\quad x\in A \quad\text{and}\quad \varphi\in L_1(G).$$

Hence, if A is right faithful i.e. the right annihilator of A is $\{0\}$, then $\overset{*}{T}$ is the unique extention of T which is a multiplier; and uniqueness holds in (3.3.3). This occurs for instance when A has a right approximate identity.

(ii) We shall not repeat the theorem concerning the Fourier transform when G is abelian as found in [9]. We shall rather treat the corresponding statement for compact groups which is a new fact.

4. $L_1(G, A)$-multipliers for compact groups

4.1. Fourier—Stieltjes transform. Let $m\in M_1(A)$. The Fourier—Stieltjes transform $\hat{m}$ of m is well known if G is abelian or if G is compact and $A=\mathbf{C}$ or $\mathbf{R}$. In fact, let G be abelian and denote its character group by $\hat{G}$. Then $\hat{m}$ is defined by the equation:

$$\hat{m}(\Gamma)=\int \bar{\Gamma}(t)\,dm(t), \quad \Gamma\in\hat{G}, \tag{4.1.1}$$

where $\bar{\Gamma}$ is the complex conjugate of Γ [9]. If G is compact and $A=\mathbf{C}$, the equality defining $\hat{m}$ becomes:

$$\langle\hat{m}(\sigma)\xi,\eta\rangle=\int\langle\bar{U}_t^{(\sigma)}\xi,\eta\rangle\,dm(t), \quad \sigma\in\Sigma, \quad (\xi,\eta)\in H_\sigma\times H_\sigma, \tag{4.1.2}$$

where Σ is the dual object of G, $U^{(\sigma)}$, a representative of the equivalent class $\sigma\in\Sigma$ and H_σ, the corresponding representation Hilbert space [5].

Now, suppose G is compact, non abelian and $A\neq\mathbf{C}$ and $\mathbf{R}$. The formula (4.1.2) is no longer meaningful because the mapping:

$$\eta\to\int\langle\bar{U}_t^{(\sigma)}\xi,\eta\rangle\,dm(t)$$

is a function from H_σ into A and as such, it is impossible to express it as a scalar product $\eta\to\langle\hat{m}(\sigma)\xi,\eta\rangle$ in general. The next lemma clarifies the situation.

4.1.3. Lemma. *The mapping* $H_\sigma\times H_\sigma\to A$:

$$(\xi,\eta)\to\int\langle\bar{U}_t^{(\sigma)}\xi,\eta\rangle\,dm(t), \quad m\in M_1(A)$$

is sesquilinear and continuous.

Proof. It is easily checked that the mapping is sesquilinear. Let us show that it is continuous.

Since $\bar{U}_t^{(\sigma)}$ is unitary for every $t\in G$, the inequality

$$\left\|\int\langle\bar{U}_t^{(\sigma)}\xi,\eta\rangle\,dm(t)\right\|_A\leqq\|\xi\|_{H_\sigma}\|\eta\|_{H_\sigma}\|m\|$$

holds, and the lemma obtains.

4.1.4. Definition. We define the Fourier—Stieltjes transform $\hat{m}$ of m, $m\in \in M_1(A)$ by the equation:

$$\hat{m}(\sigma)(\xi,\eta) \doteq \int \langle \overline{U}_t^{(\sigma)}\xi, \eta\rangle\, dm(t), \quad \sigma\in\Sigma \quad \text{and} \quad (\xi,\eta)\in H_\sigma\times H_\sigma,$$

where $\overline{U}^{(\sigma)}$ is fixed once and for all for each σ [5].

Let E and F be topological vector spaces. Denote by $\mathscr{L}(E, F)$ the space of the continuous linear mappings from E into F, by $\mathscr{B}(E\times E, F)$ the space of the continuous bilinear mappings from $E\times E$ into F and by $\mathscr{S}(E\times E, F)$ the space of the continuous sesquilinear mappings from $E\times E$ into F. We know that $\mathscr{B}(E\times E, F)$ is norm isomorphic to $\mathscr{L}(E, \mathscr{L}(E, F))$ if E and F are Banach spaces [7]. Similarly $\mathscr{S}(E\times E, F)$ is norm isomorphic to $\mathscr{L}(E, \mathscr{L}(E, F))$. Thus, if G is abelian, $\mathscr{S}(H_\sigma\times H_\sigma, A)\cong A$ for, $H_\sigma\cong \mathbf{C}$ in this case, and, if $A=\mathbf{C}$, $\mathscr{S}(H_\sigma\times H_\sigma, A)\cong\mathscr{L}(H_\sigma, H_\sigma)$ for compact groups because $\mathscr{L}(H_\sigma, \mathbf{C})\cong H_\sigma$. Hence 4.1.4 generalizes (4.1.1) and (4.1.2).

4.1.5. Injectivity of the Fourier—Stieltjes transform

Lemma. *The map* $m\to\hat{m}$ *from* $M_1(A)$ *into* $\prod_{\sigma\in\Sigma}\mathscr{S}(H_\sigma\times H_\sigma, A)$ *is one-to-one.*

Proof. Suppose $\hat{n}=\hat{m}$. Then for any $\sigma\in\Sigma$ and any $(\xi,\eta)\in H_\sigma\times H_\sigma$

$$\int_G \langle \overline{U}_t^{(\sigma)}\xi, \eta\rangle\, dn(t) = \int_G \langle \overline{U}_t^{(\sigma)}\xi, \eta\rangle\, dm(t).$$

In particular $\int_G \langle \overline{U}_t^{(\sigma)}\xi, \eta\rangle\, d(n-m)(t)=0$ for any σ, and ξ and η in an orthonormal basis of H_σ, $\sigma\in\Sigma$. According to [5] Theorem 27.39 and Remark (a) 27.8, $n-m$ is identically 0 on $\mathscr{K}(G)$. Thus $n=m$. Therefore the map is one-to-one.

4.1.6. Fourier—Stieltjes transform of a convolution

Lemma. *Assume G is compact and consider the set*

$$\hat{M}_1(A,\sigma) = \{\hat{m}(\sigma)\colon\ m\in M_1(A),\quad \sigma\in\Sigma\}.$$

Define B_σ by

$$B_\sigma(\Phi(\sigma), \hat{m}(\sigma))(\xi,\eta) = \int_G \Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi, \eta)\, dm(t), \quad \Phi\in\prod_{\sigma\in\Sigma}\mathscr{S}(H_\sigma\times H_\sigma, A),$$

$m\in M_1(A)$ *and* $(\xi,\eta)\in H_\sigma\times H_\sigma$. *Then*

(i) B_σ *is a bilinear mapping from* $\mathscr{S}(H_\sigma\times H_\sigma, A)\times\hat{M}_1(A,\sigma)$ *into* $\mathscr{S}(H_\sigma\times H_\sigma, A)$.

(ii) $\widehat{n*m}(\sigma)=B_\sigma(\hat{n}(\sigma), \hat{m}(\sigma))$, $(n,m)\in M_1(A)\times M_1(A)$.

Proof. (i) For $\sigma\in\Sigma$, $(\xi,\eta)\in H_\sigma\times H_\sigma$ and $\|\Phi(\sigma)\|=\sup\{\|\Phi(\sigma)(\alpha,\beta)\|_A\colon \|\alpha\|\leqq 1;\ \|\beta\|\leqq 1\}$, we have

$$\int_G^* \|\Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)\|_A\, dm(t) \leqq \|\Phi(\sigma)\|\,\|\xi\|\,\|\eta\| \int_G \chi_G\, d|m| = \|\Phi(\sigma)\|\,\|\xi\|\,\|\eta\|\,\|m\| < \infty.$$

Hence B_σ is well defined. Thus the continuous (hence $|m|$-measurable) function $t\to\Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)$ from G into A is $|m|$- or equivalently m-integrable. Thus $B_\sigma(\Phi(\sigma)),\ \hat{m}(\sigma)\in\mathscr{S}(H_\sigma\times H_\sigma, A)$ since

$$\left\|B_\sigma(\Phi(\sigma),\hat{m}(\sigma))(\xi,\eta)\right\|_A \leqq \int_G \|\Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)\|_A\, d|m|(t).$$

It is obvious that B_σ is bilinear.

(ii) Plainly

$$\widehat{m*n}(\sigma)(\xi,\eta) = \int \langle \overline{U}_t^{(\sigma)}\xi,\eta\rangle\, d(m*n)(t) = \iint \langle \overline{U}_{st}^{(\sigma)}\xi,\eta\rangle\, dm(s)\, dn(t) =$$
$$= \iint \langle \overline{U}_s^{(\sigma)}\overline{U}_t^{(\sigma)}\xi,\eta\rangle\, dm(s)\, dn(t) = \int \hat{m}(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)\, dn(t) = B_\sigma(\hat{m}(\sigma),\hat{n}(\sigma))(\xi,\eta).$$

Notation. We shall use the notation $\hat{m}\times\hat{n}(\sigma)$ instead of $B_\sigma(\hat{m}(\sigma),\hat{n}(\sigma))$. The second main theorem follows:

4.2. Theorem. *Suppose G is compact. Let T be a continuous linear operator: $L_1(G,A)\to L_1(G,A)$. Then*

(4.2.1) *T is a multiplier*

if and only if

(4.2.2) *there exists a $\Phi\in\prod_{\sigma\in\Sigma}\mathscr{S}(H_\sigma\times H_\sigma,\overset{*}{A})$ such that*

$$\widehat{T(f)} = \Phi\times\hat{f},\quad f\in L_1(G,A).$$

Proof. Suppose (4.2.1) and write down $T(f)=m*f$, for some $m\in M_1(G,\overset{*}{A})$ (Theorem 2.2). Then $\widehat{T(f)}=\widehat{m*f}=\hat{m}\times\hat{f}$. We obtain (4.2.2) if we put $\Phi=\hat{m}$.

Conversely, suppose (4.2.2). Then

$$\widehat{T(f*g)}(\sigma)(\xi,\eta) = \int \Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)\int f(s)g(s^{-1}t)\, ds\, dt =$$
$$= \iint \Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta) f(s) g(s^{-1}t)\, ds\, dt$$

and, for $\sigma\in\Sigma$, $\xi\in H_\sigma$ and $\eta\in H_\sigma$ we have

$$\widehat{(T(f)*g)}(\sigma)(\xi,\eta)=\int\widehat{T(f)}(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)g(t)\,dt=\iint\Phi(\sigma)(\overline{U}_{su}^{(\sigma)}\xi,\eta)f(s)g(u)\,du=$$
$$=\iint\Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)f(s)g(s^{-1}t)\,dt=\widehat{T(f*g)}(\sigma)(\xi,\eta).$$

Therefore $T(f*g)=T(f)*g$, the mapping $m\to\hat{m}$ being one-to-one. We conclude that T is a multiplier since it is supposed to be continuous.

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